Computation and Visualization of Invariant Manifolds

Master Thesis Presentation
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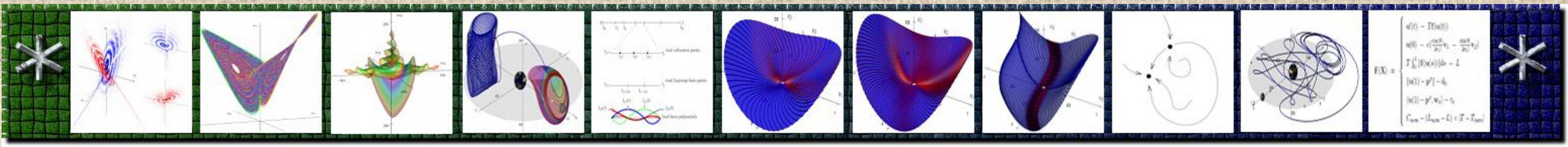


Research Objectives

- using basic dynamical systems concepts and theory
- using basic bifurcation concepts and theory
- developing visualization tools for AUTO solutions

Outline

- Introduction
- Orbit continuation
- Elementary Bifurcation Concepts
- Computing the Lorenz Manifold
- The Stability of Periodic Solutions
- The Circular Restricted Three-Body Problem
- Conclusions and Prospects

Introduction

Dynamical System

- We introduce dynamical systems and key concepts.

Differential equations and the problems AUTO solves

- The dynamical systems we deal with are defined by *n* autonomous ordinary differential equations, ODEs for short.

- AUTO can do a limited bifurcation analysis of algebraic systems.

$$\mathbf{f}(\mathbf{u}, \mathbf{p}) \;=\; 0$$

- The main algorithms in AUTO are aimed at the continuation of solutions of systems of ODEs.

$$\mathbf{u}'(t) = \mathbf{f}(\mathbf{u}(t), \mathbf{p}), \qquad \text{where } \mathbf{f}(\cdot,\cdot), \quad \mathbf{u}(\cdot) \;\in\; \mathbb{R}^n$$

Introduction

Stable and unstable invariant manifolds

In the thesis, we recall some "abstract" concepts, such as:

- Manifolds
- Invariance
- Hyperbolicity
- Stable manifold
- Unstable manifold

Introduction

Numerical methods for the computation of stable and unstable manifolds

Main tools: integration and continuation.

- Integration methods are of limited use
- Numerical continuation is an extremely effective method for computing 2D stable or unstable manifolds.
- AUTO uses continuation. The work on Doedel's AUTO was started in the mid 1970s with H. B. Keller at Caltech.

Introduction

Different approaches for computing 2D stable/unstable manifold

In the thesis, we discuss different approaches by Krauskopf and Osinga, Guckenheimer, Doedel, etc. .

Introduction

Graphical methods to visualize AUTO solutions

We implemented QTPlaut and MATPlaut. QTPlaut uses C++ with STL and QT GUI. MATPlaut uses MATLAB scripts, plotting functions and GUI.

Orbit Continuation

Stability of equilibria

We study the orbits starting from an equilibrium. We review key concepts like linearization, Jacobian matrix, eigenvalues and stability.

Orbit Continuation

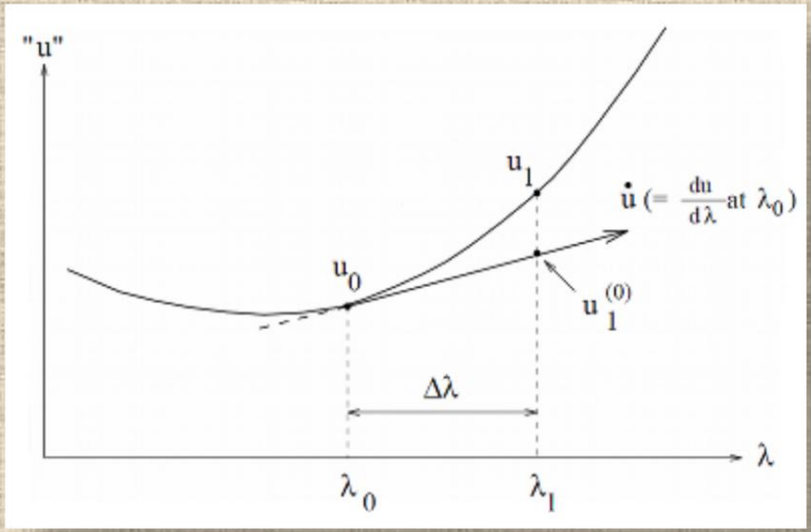
Continuation of solutions

Regular solution

- A solution $\mathbf{x_0}$ of $\mathbf{G(x) = 0}$ is regular if the n (rows) by n + 1 (columns) matrix $\mathbf{G_x(x_0)}$ has maximal rank, where $\mathbf{G}$: $\mathbf{R}^{n+1}$ -> $\mathbf{R}^n$. Near a regular solution $\mathbf{x_0}$, there exists a unique one-dimensional continuum of solution $\mathbf{x(s)}$, called a solution family or a solution branch.

Orbit Continuation

Parameter continuation



$$\begin{cases} \mathbf{G}_{\mathbf{u}}(\mathbf{u}_1^{(\nu)}, \lambda_1)\Delta\mathbf{u}_1^{(\nu)} & = & -\mathbf{G}(\mathbf{u}_1^{(\nu)}, \lambda_1), \\ \mathbf{u}_1^{(\nu+1)} & = & \mathbf{u}_1^{(\nu)} + \Delta\mathbf{u}_1^{(\nu)}, \end{cases}$$

$$\nu = 0, 1, 2, \ldots$$

$$\mathbf{u}_1^{(0)} = \mathbf{u}_0 + \Delta\lambda\dot{\mathbf{u}}_0$$

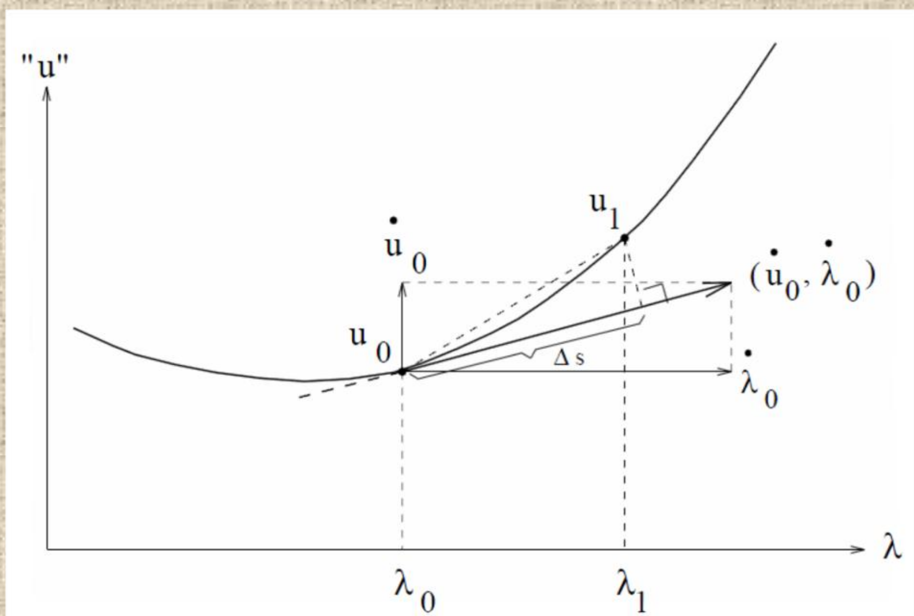
$$\mathbf{G}_{\mathbf{u}}(\mathbf{u}_1, \lambda_1)\dot{\mathbf{u}}_1 = -\mathbf{G}_\lambda(\mathbf{u}_1, \lambda_1)$$

Illustration of parameter continuation

- If $\mathbf{G}_\mathbf{u}$ is non-singular, the solution persists (locally).
- Parameter continuation will typically fail if the solution family has a fold.

Orbit Continuation

Keller's pseudo-arclength continuation



$$\begin{cases} \mathbf{G}(\mathbf{u}_1, \lambda_1) & = \mathbf{0} \\ \langle \mathbf{u}_1 - \mathbf{u}_0, \dot{\mathbf{u}}_0 \rangle + (\lambda_1 - \lambda_0)\dot{\lambda}_0 - \Delta s & = 0 \end{cases}$$

$$\begin{pmatrix} (\mathbf{G}^1_{\mathbf{u}})^{(\nu)} & (\mathbf{G}^1_{\lambda})^{(\nu)} \\ \dot{\mathbf{u}}_0^T & \dot{\lambda}_0 \end{pmatrix} \begin{pmatrix} \Delta \mathbf{u}_1^{(\nu)} \\ \Delta \lambda_1^{(\nu)} \end{pmatrix} = - \begin{pmatrix} \mathbf{G}(\mathbf{u}_1^{(\nu)}, \lambda_1^{(\nu)}) \\ \langle \mathbf{u}_1^{(\nu)} - \mathbf{u}_0, \dot{\mathbf{u}}_0 \rangle + (\lambda_1^{(\nu)} - \lambda_0)\dot{\lambda}_0 - \Delta s \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{G}^1_{\mathbf{u}} & \mathbf{G}^1_{\lambda} \\ \dot{\mathbf{u}}_0^T & \dot{\lambda}_0 \end{pmatrix} \begin{pmatrix} \dot{\mathbf{u}}_1 \\ \dot{\lambda}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}.$$

Keller's pseudo-arclength continuation allows the continuation to pass a fold.

Orbit Continuation

Keller's pseudo-arclength continuation

- AUTO uses orthogonal collocation with piecewise polynomials to solve boundary value problems. It gives very accurate results and allows adaptive mesh-selection.

- The final discretized pseudo-arclength equation

$$\sum_{j=1}^{N}\sum_{i=0}^{m}\omega_{j,i}\langle \mathbf{u}_{j-\frac{i}{m}} - (\mathbf{u}_0)_{j-\frac{i}{m}}, (\mathbf{ui}_0)_{j-\frac{i}{m}}\rangle \; + \; \langle \boldsymbol{\mu} - \boldsymbol{\mu}_0, \dot{\boldsymbol{\mu}}_0\rangle \; + \; (\lambda - \lambda_0)\dot{\lambda}_0 - \triangle s \; = \; 0$$

Orbit Continuation

Use orbit continuation to span 2D manifolds

- AUTO can also use continuation to compute solution families to initial value problem. It has a great advantage over integration of a large number of initial conditions, because the manifold described by the orbits is well covered. Even when the system we are solving has very sensitive dependence on the initial conditions, orbit continuation still gives reliable results.

Orbit Continuation

Orbit continuation to span 2D manifold -- the procedure

- The initial condition

$$\mathbf{u}(0) = \mathbf{u}_0 + \delta(\cos(\theta)\mathbf{v}_1 - \sin(\theta)\mathbf{v}_2), \quad (0 \le \theta < 2\pi)$$

- General boundary value problem

$$\mathbf{F}(\mathbf{X}) \equiv \begin{cases} \mathbf{u}'(t) \; - \; T\,\mathbf{f}(\mathbf{u}(t)) \\ \mathbf{u}(0) \; - \; \delta(\cos(\theta)\mathbf{v}_1 - \sin(\theta)\mathbf{v}_2) \\ T \; \int_0^1 \; || \, \mathbf{f}(\mathbf{u}(s)) \, || \, ds \; - L. \end{cases}$$

- The continuation procedure

$$\begin{cases} \mathbf{F}(\mathbf{X}_{i+1}) & = & \mathbf{0} \quad , \\ \langle \mathbf{X}_{i+1} - \mathbf{X}_i, \dot{\mathbf{X}}_i \rangle \; - \; \triangle s & = & 0. \end{cases}$$

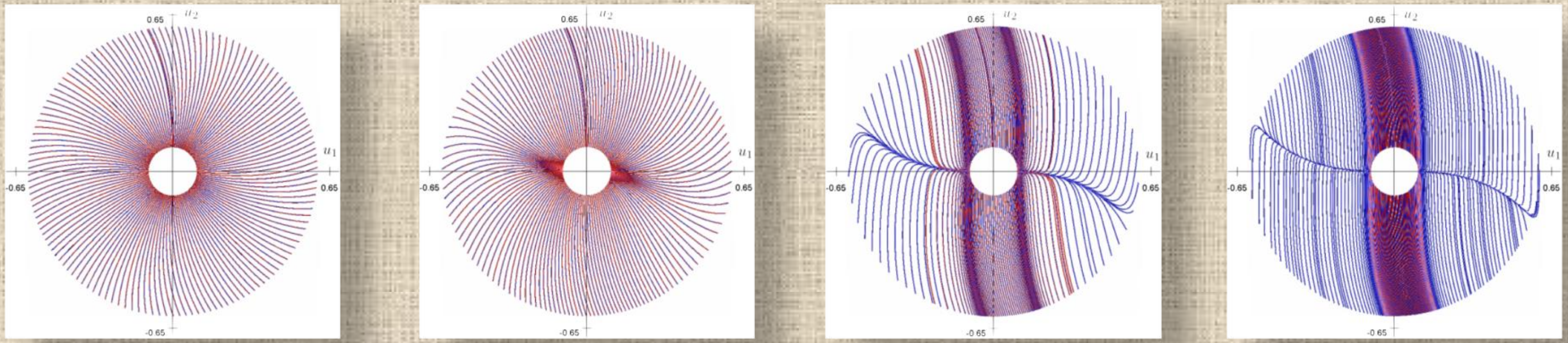
Orbit Continuation

Orbit continuation to span a 2D manifold -- example um2

- The system **u**' = **f**(**u**), where

$$\mathbf{f}(\mathbf{u}) \;=\; \begin{pmatrix} \epsilon u_1 - u_2^3 \\ u_2 + u_1^3 \end{pmatrix}$$

- Compare AUTO results with 4th-order Runge-Kutta



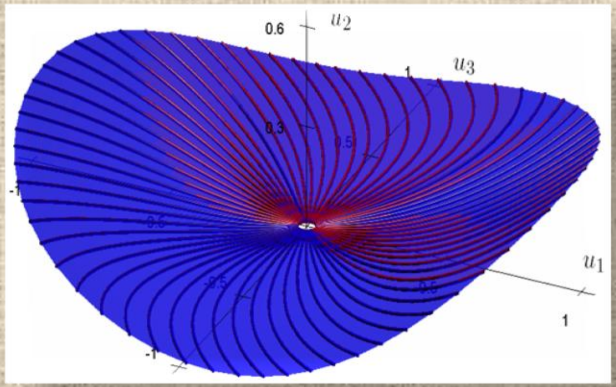
Orbit Continuation

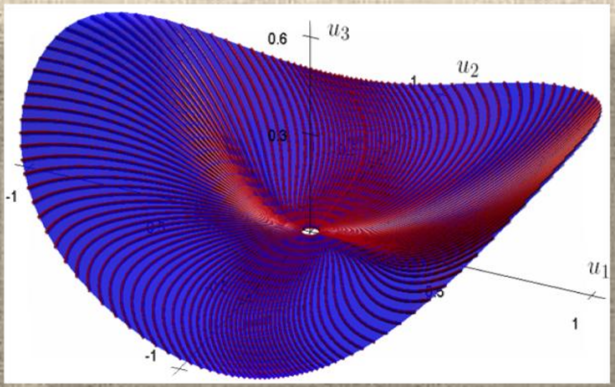
Orbit continuation to span a 2D manifold -- example um3

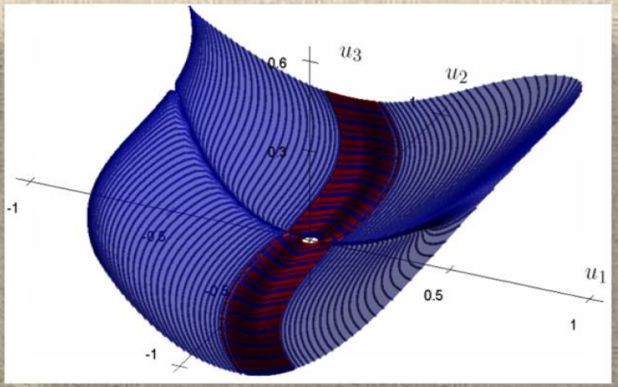
- The system **u'** = **f**(**u**), where

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} \epsilon u_1 - u_2^3 + u_3^3 \\ u_2 + u_1^3 \\ -u_3 + u_1^2 \end{pmatrix}$$

- Compare AUTO results with 4th-order Runge-Kutta



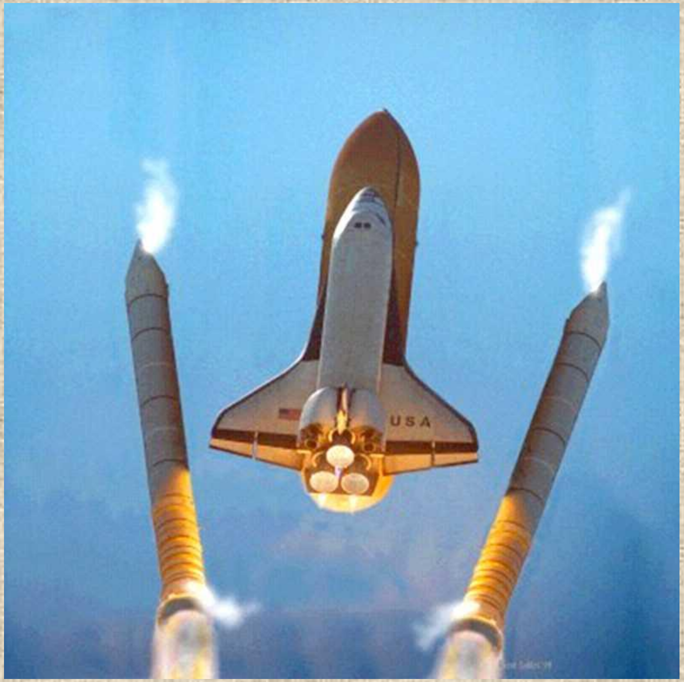




Bifurcation Concepts

The main concepts we recall and illustrate in the thesis:

- bifurcation
- bifurcation diagram
- Hopf bifurcation
- homoclinic orbit
- heteroclinic orbit



Computing the Lorenz Manifold

The system

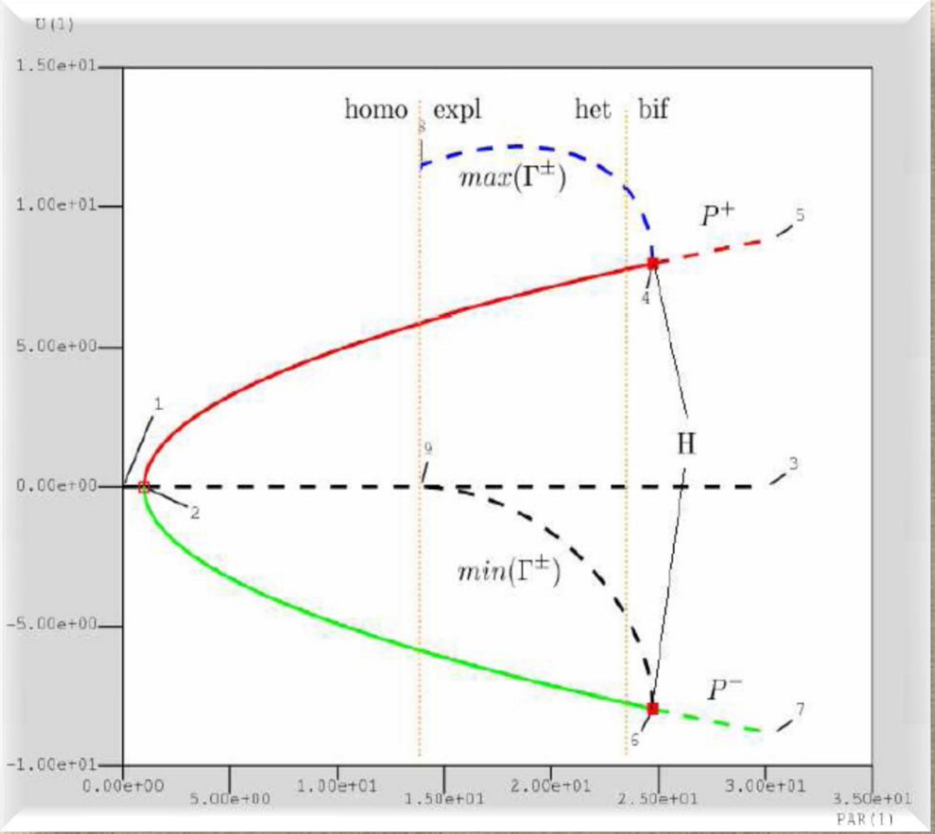
$$\begin{cases} u_1' &= \sigma(u_2 - u_1) \quad , \\ u_2' &= \rho u_1 - u_2 - u_1 u_3 \quad , \\ u_3' &= u_1 u_2 - \beta u_3 \quad . \end{cases}$$

- In the early 1960s, meteorologist Edward Lorenz started the modeling of the Earth's atmosphere. His final model involves only three elementary equations.

Computing the Lorenz Manifold

The bifurcation diagram

- It shows how the equilibria and periodic orbits are affected by parameter ρ. We also know for the typical Lorenz system , we have a 2D stable and a 1D unstable manifold at the origin. At each unstable secondary equilibrium, there are a 2D unstable manifold and a 1D stable manifold.



Computing the Lorenz Manifold

Computing the stable manifold of the origin

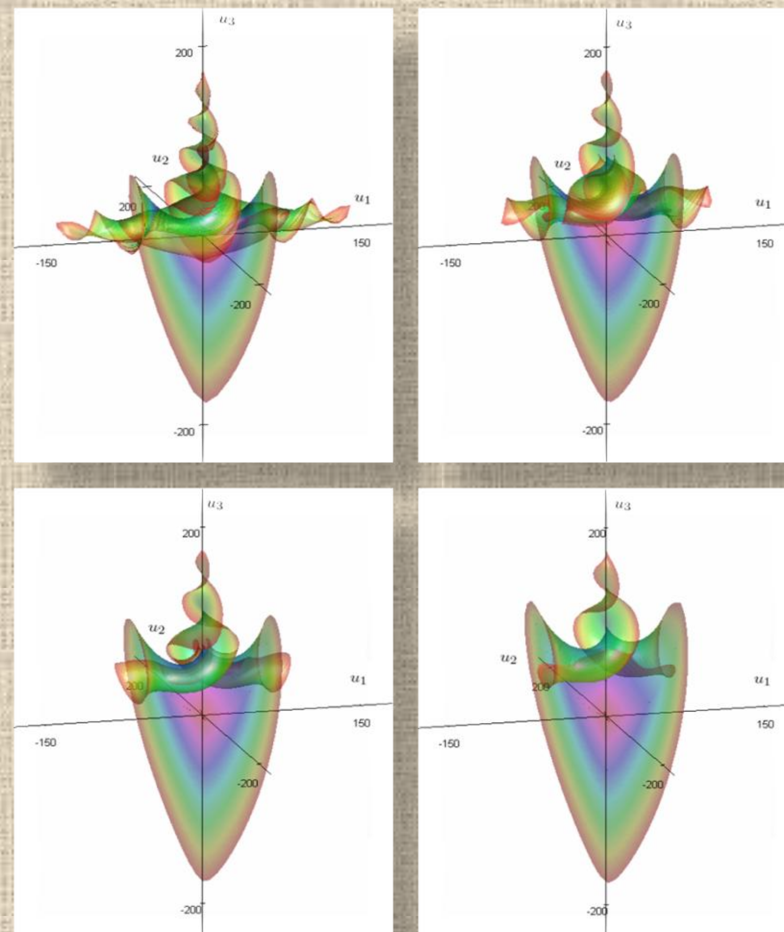
- Boundary value formulation of the continuation

$$\mathbf{F}(\mathbf{X}) \equiv \begin{cases} \mathbf{u}'(t) \; - \; T\mathbf{f}(\mathbf{u}(t)) \\ \mathbf{u}(0) \; - \; \epsilon\left(\frac{\cos(\theta)}{|\mu_1|}\mathbf{v}_1 \; - \; \frac{\sin(\theta)}{|\mu_2|}\mathbf{v}_2\right) \\ T\int_0^1 \|\mathbf{f}(\mathbf{u}(s))\| ds \; - \; L \\ C_{nom} - (L_{nom} - L) \times |T - T_{nom}| \end{cases}$$

Computing the Lorenz Manifold

Computing the stable manifold of the origin

- Results: the manifolds for ρ=15, 30, 45 and 60. They show how a planar disc ‘rolls up’ under the evolution laws.



Computing the Lorenz Manifold

Locating heteroclinic connections in the Lorenz manifold

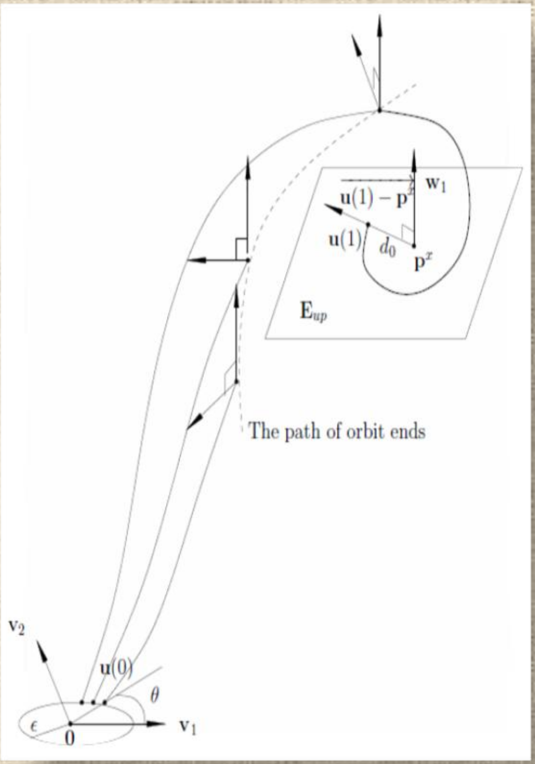
- Boundary value formulation of the continuation

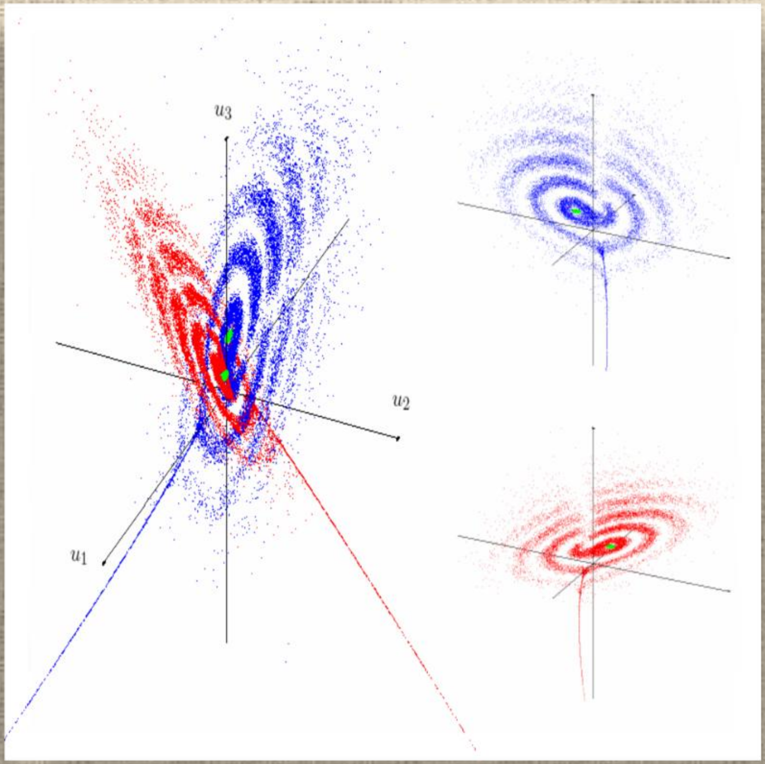
$$\mathbf{F}(\mathbf{X}) \equiv \begin{cases} \mathbf{u}'(t) \; - \; T\mathbf{f}(\mathbf{u}(t)) \\ \mathbf{u}(0) \; - \; \epsilon(\frac{\cos(\theta)}{|\mu_1|}\mathbf{v}_1 \; - \; \frac{\sin(\theta)}{|\mu_2|}\mathbf{v}_2) \\ T\int_0^1 \|\mathbf{f}(\mathbf{u}(s))\| ds \; - \; L \\ \|\mathbf{u}(1) - \mathbf{p}^x\| - d_0 \\ \langle \mathbf{u}(1) - \mathbf{p}^x, \mathbf{w}_x \rangle - \tau_x \\ C_{nom} - (L_{nom} - L) \times |T - T_{nom}| \end{cases}$$

Computing the Lorenz Manifold

Locating heteroclinic connections in the Lorenz manifold

- Results



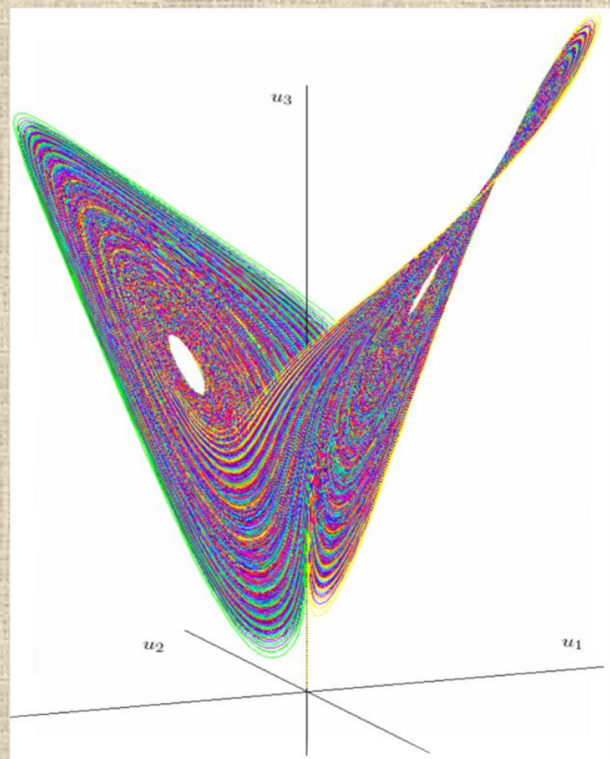


Two runs of the continuation The path of candidate orbits' end points.

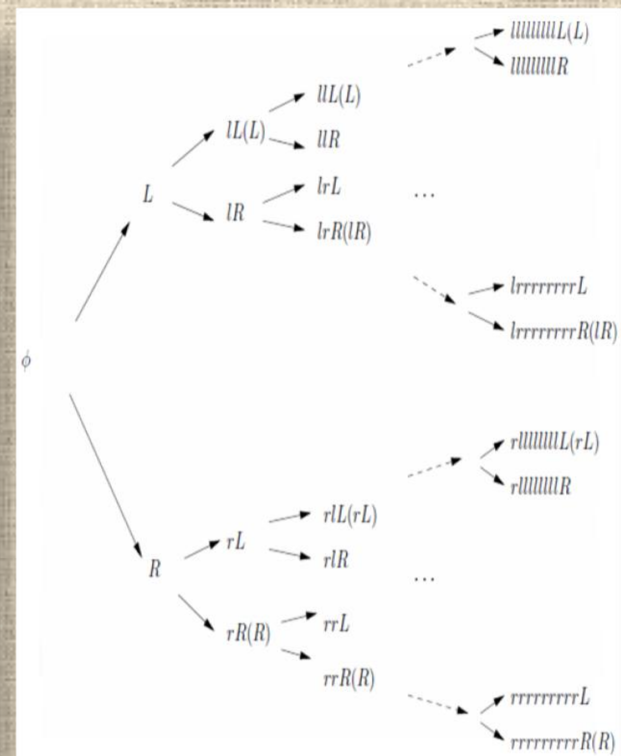
Computing the Lorenz Manifold

Symbol sequences of the heteroclinic connections

- Results 1



1024 heteroclinic orbits

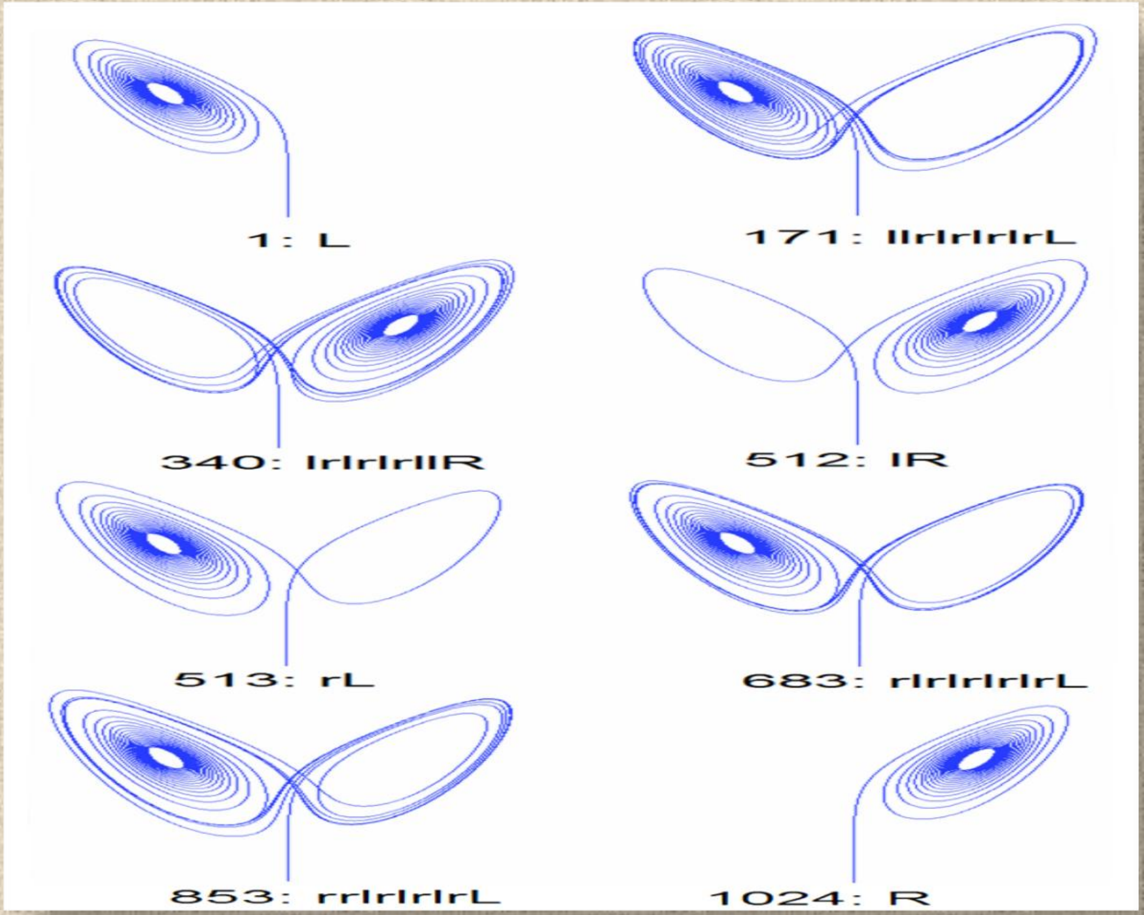


Symbol sequences ordered by length.

Computing the Lorenz Manifold

Symbol sequences of the heteroclinic connections

- Results 2



Selected heteroclinic orbits (their symbols are in the compact form).

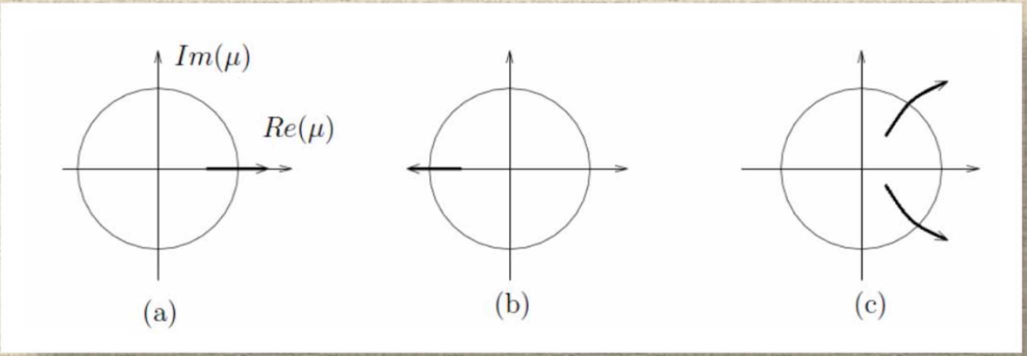
Stability of Periodic Solutions

We review and illustrate some important concepts for studying the CR3BP.

- periodic solutions of autonomous systems
- the monodromy matrix
- the Poincaré map
- Floquet multipliers
- three mechanisms of losing stability

Stability of Periodic Solutions

Three mechanisms of losing stability for periodic orbits.

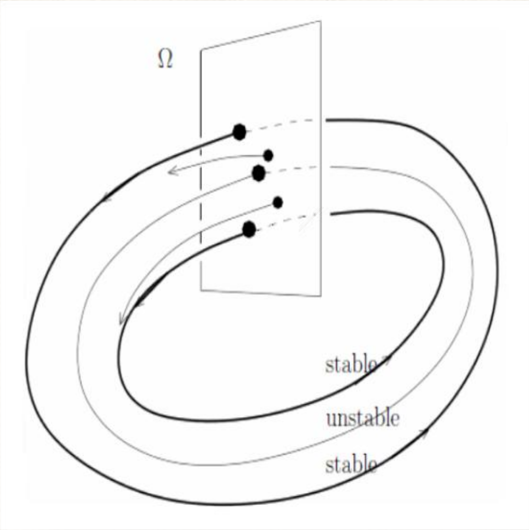


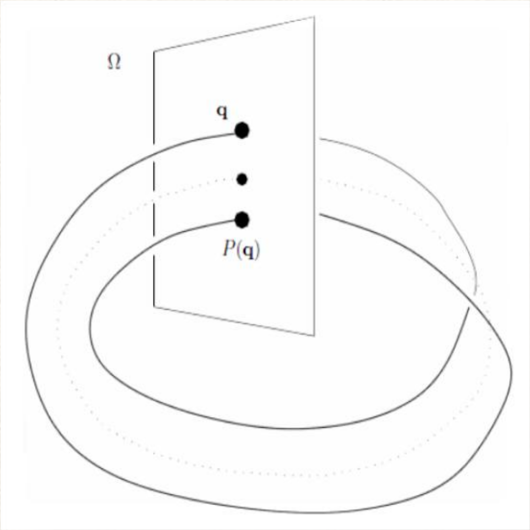
We show above the path of the critical multiplier(s)

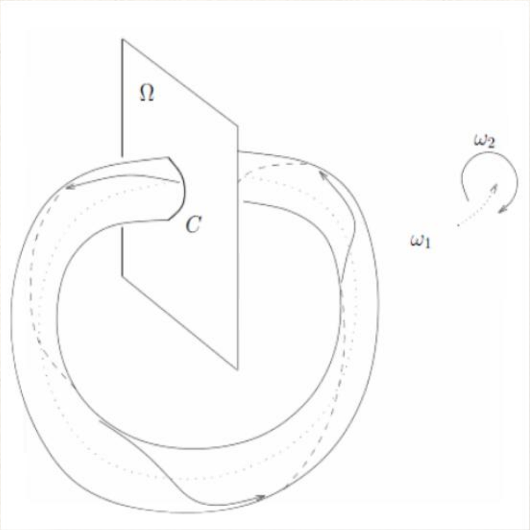
- a real multiplier passes through +1
- a real multiplier passes through -1
- a pair of complex conjugate eigenvalues cross the unit circle

Stability of Periodic Solutions

Three mechanisms of losing stability for periodic orbits.







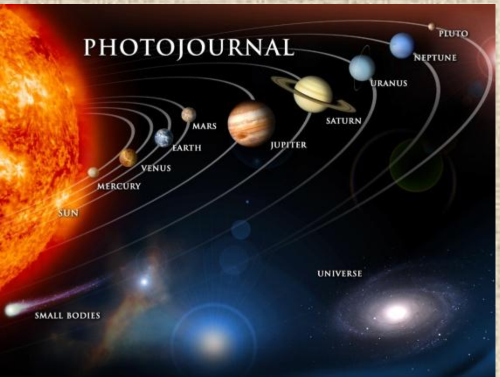
Illustrations

- close to a pitchfork bifurcation
- close to period doubling
- a torus trajectory encircles an unstable periodic orbit

The Circular Restricted Three-Body Problem

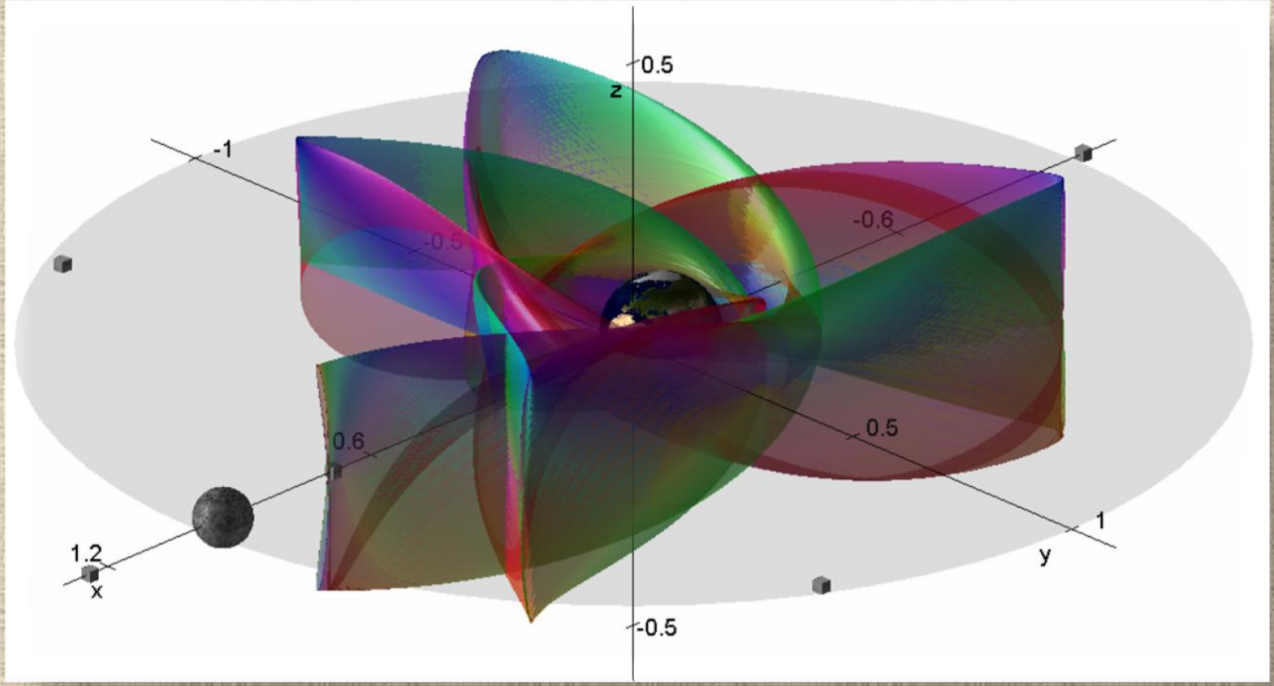
The system

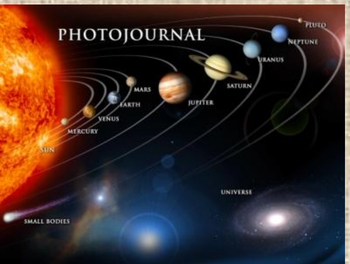
$$
\begin{aligned}
x' &= T v_x \ , \\
y' &= T v_y \ , \\
z' &= T v_z \ , \\
v_x' &= T\left(2v_y + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}\right) + \lambda v_x, \\
v_y' &= T\left(-2v_x + y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}\right) + \lambda v_y, \\
v_z' &= T\left(-\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}\right) + \lambda v_z.
\end{aligned}
$$



The Circular Restricted Three-Body Problem

Selected results

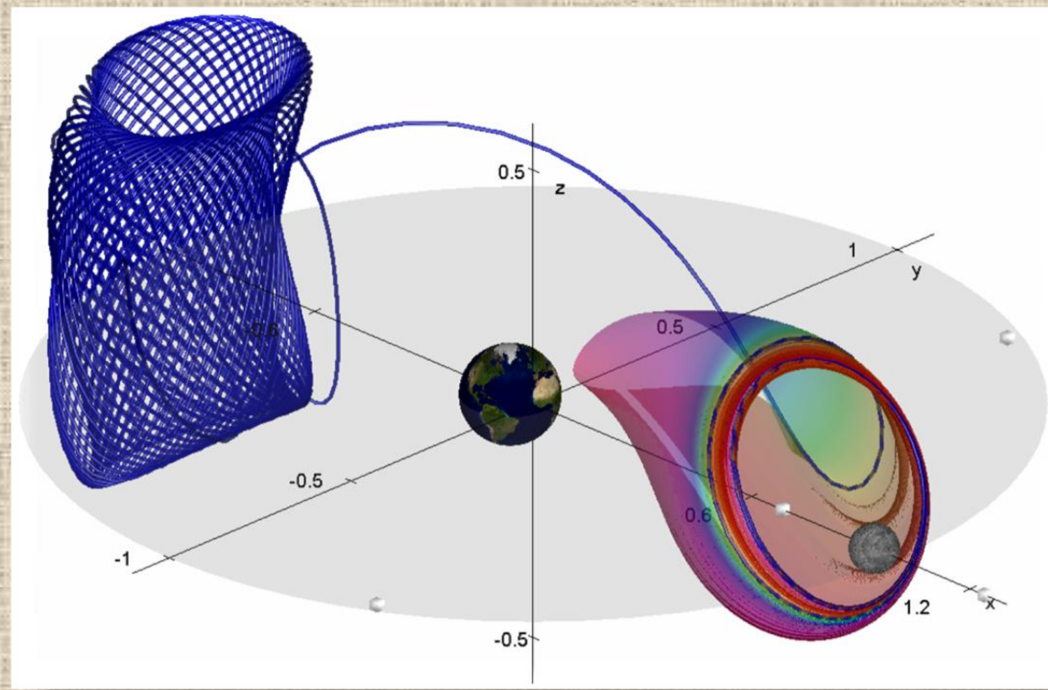


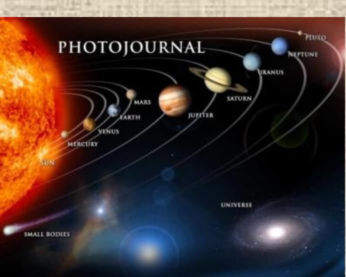


- An unstable manifold of a periodic orbit in the family $\mathbf{V}_1$ at energy -1.660129.

The Circular Restricted Three-Body Problem

Selected results

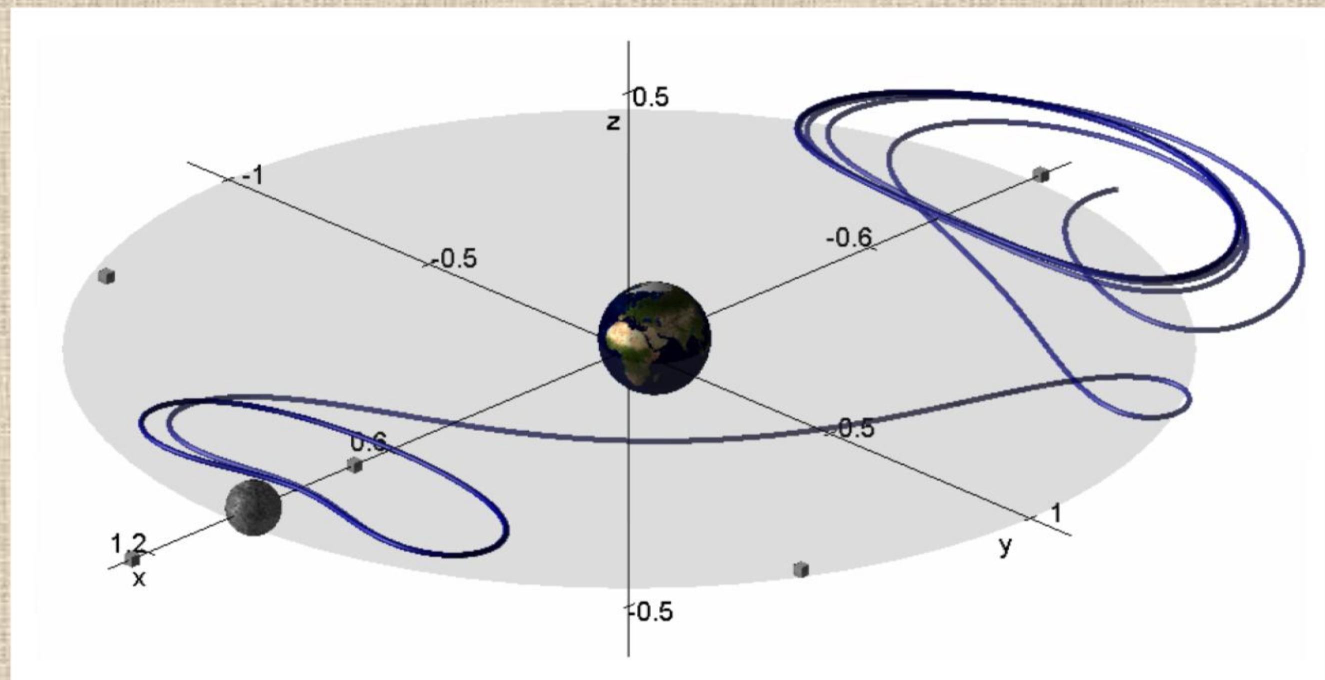


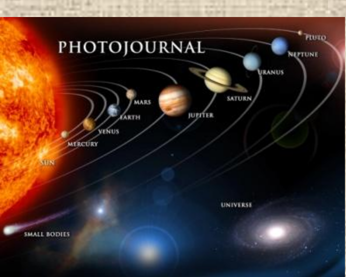


- A heteroclinic connection from a periodic orbit in family $\mathbf{H}_1$ to a torus at equilibrium $\mathbf{L}_3$ at energy -1.465585.

The Circular Restricted Three-Body Problem

Selected results





- A heteroclinic connection from a periodic orbit in family $\mathbf{L}_1$ to $\mathbf{L}_3$ at energy -1.520881.

Conclusion and Prospect

Major achievements of this thesis

- We used AUTO to study the Lorenz system: the stable manifold of the origin and the symbol sequences of heteroclinic connections.
- We used AUTO to study the CR3BP: 2D unstable manifolds of periodic orbits, and heteroclinic and homoclinic connections.

- We developed two visualization tools: QTPlaut and MATPlaut.

Conclusion and Prospect

Future research

- study the Lorenz manifold for other parameter values
- compute more CR3BP connections (different families), higher dimensional manifold
- improve AUTO: a new version GUI, extended graphics

Thanks

- The chair, my supervisor and other committee members, the audience, and NASA for some of the pictures (for decoration)!